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Abstract

Let $f:\mathbb{Z} \to \{0,1\}$ be a given function. In 1938, Morse and Hedlund observed that if the number of distinct vectors $(f(x+1),\dots,f(x+n))$, $x \in \mathbb{Z}$, called complexity, is at most n for some positive integer n, then f is periodic with period at most n. This result is best possible. Functions with low complexity have been studied to a large extent, and relations with or applications to many branches of mathematics, computer science and physics are known. In the present paper we discuss the above phenomenon in greater generality. To begin with, we observe that f is periodic if the number of distinct vectors $(f(x+a_1),\dots,f(x+a_n))$, $x \in \mathbb{Z}$, is at most n for some n and given integers $a_1 < \cdots < a_n$, but that the period cannot be bounded as a function of n only. Our main topic are multi-dimensional functions $f:\mathbb{Z}^k \to \{0,1\}$ with the property that for some n and distinct vectors $\boldsymbol{a}_1,\dots,\boldsymbol{a}_n \in \mathbb{Z}^k$, the number of distinct vectors $(f(\boldsymbol{x}+\boldsymbol{a}_1),\dots,f(\boldsymbol{x}+\boldsymbol{a}_n))$, $\boldsymbol{x} \in \mathbb{Z}^k$, is bounded by n. We show that such a function with arbitrary k is periodic if $n \leqslant 3$. For $n=4$, there are non-periodic examples which we determine completely. Finally limitations to the general periodicity principle are discussed. A conjecture for convex bodies $\{\boldsymbol{a}_1,\dots,\boldsymbol{a}_n\}$ in $\mathbb{Z}^2$ is made, and we prove it for $n \leqslant 4$. © 2000 Elsevier Science B.V. All rights reserved.

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1. Introduction

Let k be a positive integer, and let $f:\mathbb{Z}^k \to \{0,1\}$ be a given function. A non-empty set $A=\{\boldsymbol{a}_0,\boldsymbol{a}_1,\dots,\boldsymbol{a}_n\}$ of pairwise distinct vectors $\boldsymbol{a}_0,\boldsymbol{a}_1,\dots,\boldsymbol{a}_n \in \mathbb{Z}^k$ will be called a *configuration*. For each $\boldsymbol{v} \in \mathbb{Z}^k$, we define an *A-pattern* as the function $f_{\boldsymbol{v}}: A \to \{0,1\}$

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with $f_{\boldsymbol{v}}(\boldsymbol{x}) := f(\boldsymbol{v}+\boldsymbol{x})$ for all $\boldsymbol{x} \in A$, and we denote the set of all A-patterns by

$$P_f(A) := \{f_{\boldsymbol{v}} \colon \boldsymbol{v} \in \mathbb{Z}^k\}.$$

Clearly, $|P_f(A)| \leqslant 2^{n+1}$. The number $|P_f(A)|$ of distinct A-patterns is called the *A-complexity* of f.

A pattern is obviously imbedded in $\mathbb{Z}^k$ by way of the range of the corresponding $f_{\boldsymbol{v}}$. One may visualize this in the planar case by thinking of a stencil covering $\mathbb{Z}^2$ with holes only at the points $\boldsymbol{a}_0, \boldsymbol{a}_1, \ldots, \boldsymbol{a}_n \in \mathbb{Z}^2$. By shifting the stencil (without rotating it) over the plane, all A-patterns can be detected.

In order to simplify the layout, we shall also work with the equivalent definition of an A-pattern as a vector

$$(f(\boldsymbol{v}+\boldsymbol{a}_0), f(\boldsymbol{v}+\boldsymbol{a}_1), \ldots, f(\boldsymbol{v}+\boldsymbol{a}_n)) \in \{0,1\}^{n+1}$$

for some $\boldsymbol{v} \in \mathbb{Z}^k$. Then $P_f(A)$ is the set of all different vectors of this type. We like to point out that in the present paper $P_f(A)$ denotes the set of patterns generated by f and accordingly $|P_f(A)|$ is the complexity of f, whereas in the literature on Sturmian sequences P is often used for the complexity function itself.

We shall say that $f : \mathbb{Z}^k \to \{0,1\}$ is *periodic* if there is some $\boldsymbol{w} \in \mathbb{Z}^k \setminus \{\boldsymbol{0}\}$ such that $f(\boldsymbol{u}+\boldsymbol{w}) = f(\boldsymbol{u})$ for all $\boldsymbol{u} \in \mathbb{Z}^k$; of course, $\boldsymbol{0}$ denotes the k-dimensional zero vector. Then $\boldsymbol{w}$ is called a *period vector* of f; as usual, we shall simply speak of a period in case $k=1$ and consider this to be a scalar rather than a vector.

For example, let $f : \mathbb{Z}^2 \to \{0,1\}$ be given by $f(x,y)=1$ if $x>0$ and $y>0$, and $f(x,y)=0$ otherwise. Let $A = \{(0,0),(1,1),(2,0)\}$. Then f is non-periodic,

$$P_f(A) = \{(0,0,0),(0,0,1),(0,1,0),(0,1,1),(1,1,1)\},$$

and the five patterns appear in the following figure describing f around the origin.

$$\begin{array}{ccccccccccc}
0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}$$

The purpose of this paper is to examine under which conditions the following principle holds:

Periodicity Principle (PP). *Let $f : \mathbb{Z}^k \to \{0,1\}$ be a given function for a positive integer k. Let $A \subset \mathbb{Z}^k$ be a configuration. If $|P_f(A)| \leqslant |A|$, then f is periodic.*

In 1938, Morse and Hedlund [23] observed that (PP) is true for $k=1$, if the configuration is a block, i.e. of type $A=\{a, a+1, \ldots, a+n\}$ for some integers a and $n \geqslant 0$. In this case, the period can be bounded by the length $n+1$ of the block. In the next section, it will be shown that (PP) holds for arbitrary one-dimensional configurations, in particular for $k=1$, and the period can be bounded in terms of A, but not in terms of n alone. In the rest of the paper, we consider the case $k>1$.

The observation of Morse and Hedlund is fundamental. In many areas one meets non-periodic sequences which satisfy $|P_f(A)| \leqslant |A|+1$, for example in number theory (Beatty sequences, see e.g. [17, 14, 22, 30, 15]; Jacobi–Perron algorithm, see e.g. [18]), geometry (cutting sequences, see e.g. [29]), symbolic dynamics and dynamical systems theory (Sturmian sequences, see e.g. [24, 12, 9, 4, 5]), theory of automata (see e.g. [26, 2]), operations research (queueing networks, see e.g. [16, 1]), theoretical computer science (low complexity sequences, see e.g. [13, 20, 21]), and physics (quasi-crystals, see e.g. [7, 8, 28]). In many cases the concept of such sequences was generalized, sometimes still keeping it one-dimensional (for instance in [11, 25, 3]), sometimes in more dimensions (for example in discrete geometry, see [31, 6], and in quasi-crystallography, see [7, 8]). Recently, Vuillon [31] computed the complexity of two-dimensional functions on three letters which arise naturally when generalizing the concept of a cutting line to a cutting plane. In subsequent work, Berthé and Vuillon [6] (recommended for further references) show that the mentioned functions on three letters can be simplified to functions on two letters with complexity $nm+n$ for all $(n \times m)$-rectangles with positive integers n and m. In the final section of their paper they conjecture that if such a function has complexity at most nm for some n and m, then there is a periodic rational direction. In a forthcoming paper [27], the authors show that the conjecture is indeed true for $m=2$, that is, if the complexity for some $(n \times 2)$-rectangle is at most $2n$, then there is a periodic rational direction. Vuillon [31] also considered equilateral triangles instead of rectangles. Here we study the complexity of functions with respect to arbitrary configurations.

The above mentioned periodicity principle (PP) is a first guess if one wants to generalize the observation of Morse and Hedlund. Remark 1 below says that the result, if true, would be best possible. However, the main purpose of Section 3 is to exhibit some examples which illustrate that (PP) cannot be true for dimensions greater than 1 without additional assumptions. In Section 4, we prove that (PP) holds for all functions $f: \mathbb{Z}^k \to \{0,1\}$ with arbitrary k if the configuration has at most three points, while Section 5 shows that the examples of Section 3 essentially provide the only cases of configurations of four points which do not satisfy (PP). Section 6 is dedicated to a discussion of limitations to the validity of the periodicity principle (PP). In the final section we state the following conjecture (PPC):

Let $A \subseteq \mathbb{Z}^2$ be a finite set which is the restriction to $\mathbb{Z}^2$ of a convex set in $\mathbb{R}^2$. Then f is periodic if $|P_f(A)| \leqslant |A|$.

(PPC) implies the above-mentioned conjecture for rectangles. It is shown to hold for all configurations up to four points.

We conclude the introduction by making some preliminary remarks on the Periodicity Principle (PP).

Remark 1. If (PP) is true, then it is best possible. This is seen by looking at the example $f_0 : \mathbb{Z}^k \to \{0,1\}$ defined by $f_0(\mathbf{0}) := 1$ and $f_0(\boldsymbol{v}) := 0$ for all $\boldsymbol{v} \in \mathbb{Z}^k \backslash \{\mathbf{0}\}$. For any configuration $A = \{\boldsymbol{a}_0, \boldsymbol{a}_1, \ldots, \boldsymbol{a}_n\}$, the set $P_{f_0}(A)$ apparently consists of the $(n+1)$-dimensional zero vector $\mathbf{0}$ and the $n+1$ unit vectors $\boldsymbol{e}_i \in \{0,1\}^{n+1}$ $(1 \leqslant i \leqslant n+1)$ with $\boldsymbol{e}_1 = (1,0,\ldots,0), \ldots, \boldsymbol{e}_{n+1} = (0,\ldots,0,1)$. Hence $|P_{f_0}(A)| = |A| + 1$ for all A.

Remark 2. The converse of (PP) is trivially true. We even have that if f is periodic, then $|P_f(A)| \leqslant 2$ for certain arbitrarily large A's. Indeed, if a function $f : \mathbb{Z}^k \to \{0,1\}$ is periodic with period vector $\boldsymbol{w}$, say, then the configuration

$$A = \{\mathbf{0}, \boldsymbol{w}, 2\boldsymbol{w}, \ldots, n\boldsymbol{w}\}$$

apparently satisfies $P_f(A) \subseteq \{\mathbf{0}, \mathbf{1}\}$ for every positive integer n, where $\mathbf{1}$ denotes the vector whose coordinates are all 1. Thus $|P_f(A)| \leqslant 2 \leqslant |A|$.

Remark 3. For $k = 1$, the definition of periodicity for a function $f : \mathbb{Z}^k \to \{0,1\}$ is non-controversial. For higher dimensions, however, periodicity could be defined in a different, more restrictive way by requiring that for each non-zero vector $\boldsymbol{v} \in \mathbb{Z}^k$, there has to exist a positive integer l such that $l\boldsymbol{v}$ is a period vector of f. We call a function with this property *totally periodic*. The following example illustrates that total periodicity cannot be detected in general by looking at configurations with a small number of patterns:

Let $f_1 : \mathbb{Z} \to \{0,1\}$ be an arbitrary non-periodic function. Define $f_2 : \mathbb{Z}^2 \to \{0,1\}$ by setting $f_2(x,y) := f_1(x)$ for all $x, y \in \mathbb{Z}$. By construction, f_2 is not totally periodic. Yet we have only two patterns for all configurations which are subsets of $\{(0,y) \colon y \in \mathbb{Z}\}$.

Remark 4. In the sequel, the structure of $\mathbb{Z}^k$ as a lattice is important. Therefore, sublattices and cosets of sublattices (called *grids* by some authors) will be considered. Throughout this paper, any subset of a given lattice Λ which is closed under vector addition will be called a sublattice of Λ. This means that, contrary to the more frequent definition, (sub)lattices and cosets of (sub)lattices need not have full dimension.

Remark 5. Without loss of generality, we impose the following additional assumption on the configurations A in (PP):

(i) $\boldsymbol{a}_0 = \mathbf{0}$, so that $A = \{\mathbf{0}, \boldsymbol{a}_1, \ldots, \boldsymbol{a}_n\}$.

When proving (PP) by induction on $|A|$, we also assume that A is minimal so that

(ii) $|P_f(A')| > |A'|$ for all proper subsets $A' \subset A$, since otherwise the periodicity of f follows readily from (PP) applied to a smaller configuration.

(iii) $|P_f(A)| = |A|$, since otherwise $|P_f(A')| \leqslant |A'|$ for some proper subset $A' \subset A$, contrary to assumption (ii).

2. The one-dimensional case

We shall now prove (PP) in the one-dimensional case without any further restrictions. The same application of the box principle as in the classical case suffices to deal with this situation.

Theorem 1. (PP) *holds for* $k = 1$. *The period can be bounded by a function of the configuration.*

Proof. Without loss of generality, let $A = \{a_0, a_1, \dots, a_n\}$ where $0 = a_0 < a_1 < \cdots < a_n$, and $|P_f(A)| \leqslant n+1$. We assume that for all proper subsets $A' \subset A$, we have $|P_f(A')| > |A'|$, and thus $|P_f(A)| = n + 1$ (cf. Remark 5).

In case $n = 0$ we have only one A-pattern. This means that f is constant, and we are done.

Now let $n \geqslant 1$. By the minimality of A, we have for each subset $A' \subseteq A$ with $|A'| = n$

$$n < |P_f(A')| \leqslant |P_f(A)| = n + 1.$$

This implies that each of the $(n + 1)$ A'-patterns of $P_f(A')$ has a unique continuation to an A-pattern of $P_f(A)$. E.g. if b_1 and b_2 are such that

$$f(b_1 + a_i) = f(b_2 + a_i) \tag{1}$$

for $i = 0, \dots, n - 1$, then $f(b_1 + a_n) = f(b_2 + a_n)$.

Let $\bar{A} := \{1, 2, \dots, a_n\}$. Since there are at most 2^{a_n} different $\bar{A}$-patterns, there must be two integers $b_1 < b_2$ (with $b_2 - b_1 \leqslant 2^{a_n}$) such that $f(b_1 + a) = f(b_2 + a)$ for all $a \in \bar{A}$. Then $f(b_1 + a_i + 1) = f(b_2 + a_i + 1)$ for $i = 0, \dots, n - 1$, whence $f(b_1 + a_n + 1) = f(b_2 + a_n + 1)$ and therefore $f(b_1 + a) = f(b_2 + a)$ for all $a \in \bar{A} \cup \{a_n + 1\}$. Continuing in this fashion, we get

$$f(b_1 + a_n + 2) = f(b_2 + a_n + 2), \quad f(b_1 + a_n + 3) = f(b_2 + a_n + 3), \dots$$

which means that f is periodic "to the right".

By using (1) for $i = 1, \dots, n$, we obtain $f(b_1) = f(b_2)$, and from this periodicity "to the left" follows. This yields complete periodicity of f.

Remark 6. Theorem 1 clearly includes the one-dimensional case for block configurations as mentioned in the introduction. In that situation it is obvious that if the number of blocks of length n is at most n, then the corresponding function is periodic with period at most n. The preceding proof shows that if $A = \{a_0, a_1, \dots, a_n\}$ with $a_0 < a_1 < \cdots < a_n$ and $|P_f(A)| \leqslant n + 1$, then f is periodic with period at most

$2^{a_n-a_0}$. The following example shows that the period cannot be bounded in terms of the cardinality of A only.

Let N be a positive integer and define $f:\mathbb{Z}\to\{0,1\}$ by $f(x)=1$ if and only if $N\mid x$. Obviously, f is periodic with period N, but with no smaller period. Let

$$A:=\{0,a_1,a_2,\ldots,a_{n-1},N\}$$

for some $0<a_1<a_2<\cdots<a_{n-1}<N$ with N large compared to n. Apparently

$$P_f(A)\subseteq\{\mathbf{0},(1,0,\ldots,0,1),\boldsymbol{e}_2,\boldsymbol{e}_3,\ldots,\boldsymbol{e}_n\},$$

where $\boldsymbol{e}_i\in\mathbb{Z}^{n+1}$ denotes the ith unit vector. Hence $|P_f(A)|\leqslant n+1$.

As a corollary to Theorem 1, we obtain that (PP) holds for all one-dimensional configurations. By the length of a vector we mean its euclidean length.

Theorem 2. *Let* $A=\{\mathbf{0},\boldsymbol{a}_1,\ldots,\boldsymbol{a}_n\}\subset\mathbb{Z}^k$ *be such that*

$$\dim_{\mathbb{R}}(\mathbb{R}\boldsymbol{a}_1+\cdots+\mathbb{R}\boldsymbol{a}_n)=1.$$

If $f:\mathbb{Z}^k\to\{0,1\}$ *satisfies* $|P_f(A)|\leqslant|A|$, *then there is a period vector* $\boldsymbol{w}\in(\mathbb{Z}\boldsymbol{a}_1+\cdots+\mathbb{Z}\boldsymbol{a}_n)$ *of* f *whose length depends only on* A.

Proof. By our assumption, the set $\Lambda=\mathbb{Z}\boldsymbol{a}_1+\cdots+\mathbb{Z}\boldsymbol{a}_n\subseteq\mathbb{Z}^k$ is a one-dimensional sublattice of $\mathbb{Z}^k$. Hence there is some $\boldsymbol{v}\in\mathbb{Z}^k$ such that $\Lambda=\mathbb{Z}\boldsymbol{v}$. We introduce the function $f_0:\mathbb{Z}\to\{0,1\}$ by setting $f_0(x):=f(x\boldsymbol{v})$. Let $A_0=\{0,\alpha_1,\ldots,\alpha_n\}$ be the configuration defined by $\alpha_i\boldsymbol{v}=\boldsymbol{a}_i$ for $1\leqslant i\leqslant n$. Clearly, $P_f(A)\supseteq P_{f_0}(A_0)$. Hence the hypothesis of (PP) implies $|P_{f_0}(A_0)|\leqslant|P_f(A)|\leqslant n+1$. By Theorem 1, f_0 is periodic with a period which is bounded in terms of A (see Remark 6). This means that $f|_\Lambda$ has a period vector $\boldsymbol{w}\in\mathbb{Z}\boldsymbol{v}$ of bounded length.

The corresponding argument holds for each coset $\boldsymbol{u}+\Lambda$. Therefore, given any $\boldsymbol{u}\in\mathbb{Z}^k$, $f|_{\boldsymbol{u}+\Lambda}$ has a period vector $l(\boldsymbol{u})\boldsymbol{v}$, say, where $l(\boldsymbol{u})\leqslant c(A)$ for some constant $c(A)$ only depending on A. Hence f is periodic with a period vector $\lambda\boldsymbol{v}$ for some λ with $1\leqslant\lambda\leqslant\operatorname{lcm}(1,2,\ldots,c(A))$. $\square$

3. Preliminaries and two examples

We have seen in the preceding section that (PP) holds in the one-dimensional case. Now we shall present two examples that reveal that (PP) cannot be true unconditionally for any higher dimension.

First we introduce some further definitions and establish a few simple facts. Let $\boldsymbol{w}\in\mathbb{Z}^k$, and let l be a positive integer. A function $f:\mathbb{Z}^k\to\{0,1\}$ is called *$\boldsymbol{w}$-periodic of length l* if $l\boldsymbol{w}$ is a period vector of f or if $\boldsymbol{w}=\mathbf{0}$.

Lemma 1. *If $f : \mathbb{Z}^k \to \{0,1\}$ is $\boldsymbol{w}_i$-periodic of length l_i for $1 \leqslant i \leqslant d$, say, then f is $\boldsymbol{w}$-periodic of length $L := \operatorname{lcm}(l_1,\dots,l_d)$ for each $\boldsymbol{w} \in \mathbb{Z}\boldsymbol{w}_1 + \cdots + \mathbb{Z}\boldsymbol{w}_d$.*

Proof. Let $\boldsymbol{w} = a_1\boldsymbol{w}_1 + \cdots + a_d\boldsymbol{w}_d \neq \boldsymbol{0}$ for some integers a_i, and let $\boldsymbol{v} \in \mathbb{Z}^k$ be arbitrary. Since $f(\boldsymbol{u} + l_i\boldsymbol{w}_i) = f(\boldsymbol{u})$ for $1 \leqslant i \leqslant d$ and all $\boldsymbol{u} \in \mathbb{Z}^k$, we obtain

$$\begin{aligned} f(\boldsymbol{v} + L\boldsymbol{w}) &= f(\boldsymbol{v} + La_1\boldsymbol{w}_1 + \cdots + La_d\boldsymbol{w}_d) \\ &= f(\boldsymbol{v} + La_1\boldsymbol{w}_1 + \cdots + La_{d-1}\boldsymbol{w}_{d-1}) \\ &= \cdots = f(\boldsymbol{v}), \end{aligned}$$

and Lemma 1 is proven. $\square$

It follows that the set of all periods of $f : \mathbb{Z}^k \to \{0,1\}$ is a sublattice $\Lambda_f \subseteq \mathbb{Z}^k$, say, spanned by some maximal set of linearly independent periods of f. We shall call Λ_f the *period lattice of f*, and $\dim \Lambda_f$ the *period dimension*. We clearly have: $\dim \Lambda_f \geqslant 1$ if and only if f is periodic, and $\dim \Lambda_f = k$ if and only if f is totally periodic. For any set $\Lambda \subseteq \mathbb{Z}^k$, we shall say that f is *Λ-periodic of length l* if $l\boldsymbol{w}$ is a period vector of f for all $\boldsymbol{w} \in \Lambda$. If $l\boldsymbol{w}$ is a period vector of f for all $\boldsymbol{w} \in \mathbb{Z}^k$, then f is called *totally periodic of length l*.

In order to construct counterexamples to (PP), we make use of the following process. Let $f_i : \mathbb{Z}^k \to \{0,1\}$ be arbitrary functions for $1 \leqslant i \leqslant n$. We define a function $f : \mathbb{Z}^k \to \{0,1\}$ by setting

$$f(nx_1 + i, x_2, \dots, x_k) := f_i(x_1, \dots, x_k)$$

for $x_1, \dots, x_k \in \mathbb{Z}$ and $1 \leqslant i \leqslant n$ and call f the *slice function* of $f_1, \dots, f_n$.

Lemma 2. *Let $f_i : \mathbb{Z}^k \to \{0,1\}$ be arbitrary functions with corresponding period lattices Λ_{f_i} for $1 \leqslant i \leqslant n$, and let f be the slice function of $f_1, \dots, f_n$. Then*

$$\Lambda_f^{(n)} := \{(w_1, nw_2, \dots, nw_k) : (w_1, \dots, w_k) \in \Lambda_f\}$$

is a sublattice of $\bigcap_{i=1}^n \Lambda_{f_i}$. In particular, if $\bigcap_{i=1}^n \Lambda_{f_i} = \{\boldsymbol{0}\}$, then f is not periodic.

Proof. Let $\boldsymbol{w} = (w_1, \dots, w_k)$ be a period vector of f. Then $n\boldsymbol{w}$ is also a period vector of f. Hence $(w_1, nw_2, \dots, nw_k)$ is a period vector of every f_i $(1 \leqslant i \leqslant n)$. $\square$

For given vectors $\boldsymbol{w}_1, \dots, \boldsymbol{w}_n \in \mathbb{Z}^k$, we denote by $\Lambda(\boldsymbol{w}_1, \dots, \boldsymbol{w}_n)$ the lattice $\mathbb{Z}\boldsymbol{w}_1 + \cdots + \mathbb{Z}\boldsymbol{w}_n$. The following example shows that (PP) is not true for any dimension greater than 1.

Example 1. Let $k \geqslant 1$ be given. For $1 \leqslant i \leqslant k$ let $f_i : \mathbb{Z}^k \to \{0,1\}$ be defined by $f_i(x_1, \dots, x_k) = 1$ if and only if $x_i \geqslant 0$. Clearly, we have the period lattices

$$\Lambda_{f_i} = \Lambda(\boldsymbol{e}_1, \dots, \boldsymbol{e}_{i-1}, \boldsymbol{e}_{i+1}, \dots, \boldsymbol{e}_k)$$

for $1 \leqslant i \leqslant k$, where $\boldsymbol{e}_i$ denotes the ith k-dimensional unit vector. Therefore,

$$\bigcap_{i=1}^{k} \Lambda_{f_i} = \{\mathbf{0}\}.$$

By Lemma 2, this implies that the slice function f of $f_1, \dots, f_k$ is not periodic.

We define the configuration

$$A := \{(\alpha_1, \alpha_2, \dots, \alpha_k)\colon\ \alpha_i \in \{0, 1\}\}.$$

Apparently, $|A| = 2^k$. On A, we have for $1 \leqslant i \leqslant k$,

$$P_{f_i}(A) = \{\mathbf{0}, \mathbf{1}, \boldsymbol{a}_i\} \subset \{0, 1\}^{2^k},$$

where $\mathbf{1}$ denotes the vector with all coordinates equal to 1, and where $\boldsymbol{a}_i$ is a specific vector with exactly 2^{k-1} coordinates 1. More precisely, A can be ordered in such a way that $\boldsymbol{a}_i$ represents the sequence of the ith digits of the binary expansion of the integers $0, 1, 2, \dots, 2^k - 1$. We have $\boldsymbol{a}_i \neq \boldsymbol{a}_j$ for $1 \leqslant i < j \leqslant k$. Now let

$$A_k := \{(\kappa, \alpha_2, \dots, \alpha_k)\colon\ \kappa \in \{0, k\},\ \alpha_i \in \{0, 1\}\}.$$

Then the slice function f of $f_1, \dots, f_k$ satisfies $|P_f(A_k)| = k + 2$.

For $k = 2$ we have $\boldsymbol{a}_1 = (0, 1, 0, 1)$ and $\boldsymbol{a}_2 = (0, 0, 1, 1)$ (under suitable arrangement of A), and the slice function looks around the origin like

$$\begin{array}{cccccccccccc}
0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1\\
0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1\\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0
\end{array}$$

Conclusion. *Let k be a given positive integer. Then there is a non-periodic function $f : \mathbb{Z}^k \to \{0, 1\}$ and a configuration $A \subset \mathbb{Z}^k$ with $|A| = 2^k$ such that $|P_f(A)| = k + 2$. Thus* (PP) *is wrong for every dimension $k \geqslant 2$.*

Remark 7. We have shown that the number of patterns can be of logarithmic order of the cardinality of the configuration for a non-periodic function f.

We discuss another example. Examples 1 and 2 provide the only exceptions to (PP) for $|A| = 4$ (cf. Theorem 4).

Example 2. We define three functions $f_i : \mathbb{Z}^3 \to \{0, 1\}$, $i = 1, 2, 3$, in the following way: For every $y \in \mathbb{Z}$ let $f_1(x, 2y, 0)$ be alternating in $x \in \mathbb{Z}$ in such a way that f_1 is not $\boldsymbol{e}_2$-periodic. (A possible choice is $f_1(x, 2y, 0) = 0$ if x is odd and $y = 0$ and if x is even and $y \neq 0$, and $f_1(x, 2y, 0) = 1$ otherwise.) Put $f_1(x, 2y + 1, 0) = 0$ for all x and y, and $f_1(x, y, z) = f_1(x - z, y + z, 0)$ for all x, y and $z \neq 0$. Hence

$$\Lambda_{f_1} = \Lambda(2\boldsymbol{e}_1, \boldsymbol{e}_1 - \boldsymbol{e}_2 + \boldsymbol{e}_3). \tag{2}$$

Note that for every pair x, y exactly one among $f_1(x, y, 0)$, $f_1(x, y+1, 0)$, $f_1(x+1, y, 0)$, $f_1(x-1, y+1, 0)$ equals 1, whence the configuration $A_1 := \{\mathbf{0}, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ satisfies

$$P_{f_1}(A_1) = \{(1,0,0,0),(0,1,0,0),(0,0,1,0),(0,0,0,1)\}.$$

Similarly, we let $f_2(2x, y, 0)$ be alternating in y for fixed x such that f_2 is not $\mathbf{e}_1$-periodic, $f_2(2x+1, y, 0) = 0$ for all x and y, and $f_2(x, y, z) = f_2(x+z, y-z, 0)$ for all x, y, z. Then $P_{f_2}(A_1) = P_{f_1}(A_1)$, and

$$\Lambda_{f_2} = \Lambda(2\mathbf{e}_2, \mathbf{e}_1 - \mathbf{e}_2 - \mathbf{e}_3). \tag{3}$$

Further we require $f_3(x, 2y-x, 0)$ to be alternating in x for fixed y such that f_3 is not $(\mathbf{e}_1+\mathbf{e}_2)$-periodic, $f_3(x, 2y+1-x, 0) = 0$ for all x and y, and $f_3(x, y, z) = f_3(x+z, y-z, 0)$ for all x, y, z. Then $P_{f_3}(A_1) = P_{f_1}(A_1)$, and

$$\Lambda_{f_3} = \Lambda(2(\mathbf{e}_1 - \mathbf{e}_2), \mathbf{e}_1 - \mathbf{e}_2 - \mathbf{e}_3). \tag{4}$$

Let $f : \mathbb{Z}^3 \to \{0,1\}$ denote the slice function of f_1, f_2, f_3. Put $A_3 := \{\mathbf{0}, 3\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$. By construction,

$$P_f(A_3) = P_{f_1}(A_1) = P_{f_2}(A_1) = P_{f_3}(A_1).$$

It follows from (2)–(4) that $\bigcap_{i=1}^{3} \Lambda_{f_i} = \{\mathbf{0}\}$. Hence f is not periodic by Lemma 2. Since $|A_3| = |P_f(A_3)| = 4$, this contradicts (PP).

4. Configurations up to three points

In this section we shall prove

Theorem 3. (PP) *is true for arbitrary* $f : \mathbb{Z}^k \to \{0,1\}$ *and all configurations* A *with* $|A| \leqslant 3$.

In order to examine (PP) for configurations up to three points (and later on for configurations of four points), we shall first prove it for special configurations. To this end, we consider for given non-negative integers $n_0 < n_1 < \cdots < n_t \leqslant k$ the configuration

$$E_k(n_0, n_1, \ldots, n_t) := \{\mathbf{e}_{n_0}, \mathbf{e}_{n_1}, \ldots, \mathbf{e}_{n_t}\},$$

where $\mathbf{e}_0$ denotes the k-dimensional zero vector, and $\mathbf{e}_1, \ldots, \mathbf{e}_n$ denote the k-dimensional unit vectors, respectively. The set of all $E_k(n_0, n_1, \ldots, n_t)$-patterns for a given $f : \mathbb{Z}^k \to \{0,1\}$ will be written as

$$P_f(n_0, n_1, \ldots, n_t) := P_f(E_k(n_0, n_1, \ldots, n_t)).$$

We shall consider (PP) for small configurations of this type. It takes additional effort to deduce the corresponding part of (PP) in general. This will be done by induction on the cardinality of the configuration for configurations up to three points (and in the next section for four points).

We start the proof of Theorem 3 by classifying the functions according to their $E_n(0,1,2,\dots,n)$-patterns for $n \leqslant 2$.

Proposition 1. *Let $f:\mathbb{Z}^k \to \{0,1\}$ be a given function with $|P_f(0,n)| \leqslant 2$ for some positive integers $n \leqslant k$. Then f is $\boldsymbol{e}_n$-periodic of length* 2.

Proof. This follows from Theorem 2 and Remark 6. It is also an immediate consequence of the fact that if f is not constant, then $P_f(0,n)=\{(0,1),(1,0)\}$.

Proposition 2. *Let $f:\mathbb{Z}^k \to \{0,1\}$ be a given function with $|P_f(0,n_1,n_2)| \leqslant 3$ for some positive integers $n_1 < n_2 \leqslant k$. Then f is $\Lambda(\boldsymbol{e}_{n_1},\boldsymbol{e}_{n_2})$-periodic of length at most* 3 *except for the following two cases* (*up to interchange of* 0 *and* 1 *or permutation of coordinates*):

$$P_f(0,n_1,n_2)=\{(0,0,0),(0,0,1),(1,1,0)\}$$

or

$$P_f(0,n_1,n_2)=\{(0,0,0),(0,0,1),(1,1,1)\},$$

where the corresponding functions f are $\boldsymbol{e}_{n_1}$-periodic. More generally, the interchange of 0 *and* 1 *and/or the permutation of coordinates leads to a function which is $\boldsymbol{e}_{n_1}$-periodic or $\boldsymbol{e}_{n_2}$-periodic or $(\boldsymbol{e}_{n_1}-\boldsymbol{e}_{n_2})$-periodic of length* 1.

Proof. Without loss of generality, let $k=2$, thus $n_1=1$, $n_2=2$. We first assume that $|P_f(0,1,2)| \leqslant 2$. Then trivially $|P_f(0,n)| \leqslant 2$ for $n=1,2$. By Proposition 1, f is $\boldsymbol{e}_n$-periodic of length 2 for $n=1,2$. Lemma 1 implies total periodicity of length 2 which proves the assertion.

It remains to consider the case $|P_f(0,1,2)|=3$. If there is some $E' \subset E_2(0,1,2)$ with $|E'|=|P_f(E')|=1$, then f is constant. Next assume that there is some $E' \subset E_2(0,1,2)$ with $|E'|=|P_f(E')|=2$. Without loss of generality, let $E'=E_2(0,1)$. By Proposition 1, f is $\boldsymbol{e}_1$-periodic of length 2. This implies that

$$P_f(E')=\{(0,0),(1,1)\} \quad \text{or} \quad P_f(E')=\{(0,1),(1,0)\}.$$

Without loss of generality (that is up to interchange of 0 and 1 or permutation of coordinates), we may thus assume that

$$P_f(0,1,2)=\{(0,0,\varepsilon_1),(0,0,\varepsilon_2),(1,1,\varepsilon_3)\} \tag{5}$$

or

$$P_f(0,1,2)=\{(0,1,\varepsilon_1),(0,1,\varepsilon_2),(1,0,\varepsilon_3)\} \tag{6}$$

for some $\varepsilon_1,\varepsilon_2,\varepsilon_3 \in \{0,1\}$. In case (5), we have

$$P_f(0,1,2)=\{(0,0,0),(0,0,1),(1,1,0)\}$$

or

$$P_f(0,1,2)=\{(0,0,0),(0,0,1),(1,1,1)\}.$$

These patterns are imbedded in $\mathbb{Z}^2$ and will be denoted in the obvious fashion by

$$\begin{matrix} 0 & \\ 0 & 0 \end{matrix} \qquad \begin{matrix} 1 & \\ 0 & 0 \end{matrix} \qquad \begin{matrix} 0 & \\ 1 & 1 \end{matrix} \tag{7}$$

and

$$\begin{matrix} 0 & \\ 0 & 0 \end{matrix} \qquad \begin{matrix} 1 & \\ 0 & 0 \end{matrix} \qquad \begin{matrix} 1 & \\ 1 & 1 \end{matrix} \tag{8}$$

respectively. The functions corresponding to (7) are those with constant horizontal lines (in the direction of $\boldsymbol{e}_1$) such that there are no two consecutive horizontal lines of 1's, but at least one such line. All such functions are $\boldsymbol{e}_1$-periodic of length 1. The functions corresponding to (8) are those with constant horizontal lines (in the direction of $\boldsymbol{e}_1$) such that we have an upper half plane of 1's and a lower half plane of 0's. Again all such functions are $\boldsymbol{e}_1$-periodic of length 1.

Similarly (2) leads to

$$\begin{matrix} 0 & \\ 0 & 1 \end{matrix} \qquad \begin{matrix} 1 & \\ 0 & 1 \end{matrix} \qquad \begin{matrix} 0 & \\ 1 & 0 \end{matrix} \tag{9}$$

and

$$\begin{matrix} 0 & \\ 0 & 1 \end{matrix} \qquad \begin{matrix} 1 & \\ 0 & 1 \end{matrix} \qquad \begin{matrix} 1 & \\ 1 & 0 \end{matrix} \tag{10}$$

respectively. Trying to extend the first pattern of (9) to the right necessarily yields

$$\begin{matrix} 0 & 0 & \\ 0 & 1 & 0 \end{matrix} \tag{11}$$

This, however, cannot be extended upwards. Hence (9) does not occur. Similarly, the second pattern of (10) leads to

$$\begin{matrix} 1 & 1 & \\ 0 & 1 & 0 \end{matrix} \tag{12}$$

which is impossible as well. Therefore, (10) does not occur either.

We are left with the situation that

$$|P_f(E')|>|E'| \tag{13}$$

for all proper subsets $E'\subset E_2(0,1,2)$. Let

$$P_f(0,1,2)=\{(\varepsilon_{i,0},\varepsilon_{i,1},\varepsilon_{i,2}):\ i=1,2,3\},$$

say, with three distinct elements. By (13), applied to $E'=E_2(0)$, we may assume without loss of generality that $\varepsilon_{1,0}=1$, $\varepsilon_{2,0}=\varepsilon_{3,0}=0$. Using (13) also for $E'=E_2(0,1)$, we have without loss of generality $\varepsilon_{2,1}=0$, $\varepsilon_{3,1}=1$. We distinguish the four possible cases for $\varepsilon_{1,1}$, $\varepsilon_{1,2}$ and obtain

$$(\varepsilon_{1,0},\varepsilon_{1,1},\varepsilon_{1,2})\in\{(1,0,0),(1,0,1),(1,1,0),(1,1,1)\},$$

$$(\varepsilon_{2,0},\varepsilon_{2,1},\varepsilon_{2,2})=(0,0,\varepsilon_{2,2}),\quad (\varepsilon_{3,0},\varepsilon_{3,1},\varepsilon_{3,2})=(0,1,\varepsilon_{3,2}).$$

For $(\varepsilon_{1,0},\varepsilon_{1,1},\varepsilon_{1,2})=(1,0,0)$, we get from (13) for $E'=E_2(1,2)$ that $\varepsilon_{2,2}=1$, and then from (13) for $E'=E_2(0,2)$ that $\varepsilon_{3,2}=0$. In a similar fashion, (13) applied to appropriate sets E' yields $\varepsilon_{2,2}=0$ and $\varepsilon_{3,2}=1$ for $(\varepsilon_{1,0},\varepsilon_{1,1},\varepsilon_{1,2})=(1,0,1)$, $\varepsilon_{3,2}=1$ and $\varepsilon_{2,2}=0$ for $(\varepsilon_{1,0},\varepsilon_{1,1},\varepsilon_{1,2})=(1,1,0)$, and finally $\varepsilon_{3,2}=0$ and $\varepsilon_{2,2}=1$ for $(\varepsilon_{1,0},\varepsilon_{1,1},\varepsilon_{1,2})=(1,1,1)$. We end up with four possible sets of patterns, namely

$$P_f(0,1,2)=\{(1,0,0),(0,0,1),(0,1,0)\}, \tag{14}$$

or

$$P_f(0,1,2)=\{(1,0,1),(0,0,0),(0,1,1)\}, \tag{15}$$

or

$$P_f(0,1,2)=\{(1,1,0),(0,0,0),(0,1,1)\},$$

or

$$P_f(0,1,2)=\{(1,1,1),(0,0,1),(0,1,0)\}.$$

The last three cases are apparently equivalent which is seen by interchanging 0 and 1 or by renumbering coordinates. Therefore, it suffices to deal with (14) and (15).

To begin with, let us consider (15). In the notation used earlier, we have the three two-dimensional patterns

$$\begin{array}{ll} 1 & \\ 1 & 0 \end{array}\qquad\qquad \begin{array}{ll} 0 & \\ 0 & 0 \end{array}\qquad\qquad \begin{array}{ll} 1 & \\ 0 & 1 \end{array}$$

Continuing the third pattern to the right, i.e. in the direction of $\boldsymbol{e}_1$, we necessarily obtain

$$\begin{array}{lll} 1 & 1 & \\ 0 & 1 & 0 \end{array}$$

but there is no pattern $(1,1,\varepsilon)$. Hence (15) does not occur.

The final case to consider is (14), that is

$$\begin{array}{ll} 0 & \\ 1 & 0 \end{array}\qquad\qquad \begin{array}{ll} 1 & \\ 0 & 0 \end{array}\qquad\qquad \begin{array}{ll} 0 & \\ 0 & 1 \end{array}$$

Obviously, every 1 is surrounded by 0's in the following manner:

$$\begin{array}{lll} 0 & 0 & \\ 0 & 1 & 0 \\ & 0 & 0 \end{array}$$

Using the three patterns we have, it follows that in the positions ε_i with $1 \leqslant i \leqslant 9$ of

$$\begin{array}{cccc} \varepsilon_5 & \varepsilon_6 & \varepsilon_7 & \varepsilon_9 \\ \varepsilon_4 & 0 & 0 & \varepsilon_8 \\ \varepsilon_3 & 0 & 1 & 0 \\ \varepsilon_2 & \varepsilon_1 & 0 & 0 \end{array}$$

we must have $\varepsilon_1 = 1$, $\varepsilon_2 = 0$, $\varepsilon_3 = 0$, $\varepsilon_4 = 1$, $\varepsilon_5 = 0$, $\varepsilon_6 = 1$, $\varepsilon_7 = 0$, $\varepsilon_8 = 1$, $\varepsilon_9 = 0$. This leads to a function f with 1's exactly at every third diagonal in the direction of $\boldsymbol{e}_1 + \boldsymbol{e}_2$. Such an f is certainly totally periodic of length 3. $\square$

Proof of Theorem 3. Let $A = \{\boldsymbol{0}, \boldsymbol{a}_1, \boldsymbol{a}_2\} \subseteq \mathbb{Z}^k$. If $\dim_{\mathbb{R}} (\mathbb{R}\boldsymbol{a}_1 + \mathbb{R}\boldsymbol{a}_2) \leqslant 1$, in particular for $|A| \leqslant 2$, the assertion follows from Theorem 2. We may thus assume that $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$ are linearly independent.

We define the two-dimensional sublattice $\Lambda := \mathbb{Z}\boldsymbol{a}_1 + \mathbb{Z}\boldsymbol{a}_2$ of $\mathbb{Z}^k$. Let $h : \mathbb{Z}^2 \to \Lambda$ be the linear map defined by $h(x_1, x_2) := x_1\boldsymbol{a}_1 + x_2\boldsymbol{a}_2$ for $x_1, x_2 \in \mathbb{Z}$. The linear independence of $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$ implies that h is bijective. Given some $\boldsymbol{u} \in \mathbb{Z}^k$, we define the function $f_{\boldsymbol{u}} : \mathbb{Z}^2 \to \{0, 1\}$ by $f_{\boldsymbol{u}}(\boldsymbol{x}) := f(\boldsymbol{u} + h(\boldsymbol{x}))$ for $\boldsymbol{x} \in \mathbb{Z}^2$. The hypothesis of (PP) implies that $|P_{f_{\boldsymbol{u}}}(0, 1, 2)| \leqslant 3$ for each $\boldsymbol{u} \in \mathbb{Z}^k$. Therefore, by Proposition 2, $f_{\boldsymbol{u}}$ is $\boldsymbol{w}$-periodic of length $l(\boldsymbol{u}) \leqslant 3$ for each $\boldsymbol{u}$, where $\boldsymbol{w} \in \{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_1 - \boldsymbol{e}_2\}$ does not depend on $\boldsymbol{u}$. Consequently, each $f_{\boldsymbol{u}}$ is $\boldsymbol{w}$-periodic of length $\mathrm{lcm}(1, 2, 3) = 6$. Hence $f|_{\boldsymbol{u}+\Lambda}$ is $h(\boldsymbol{w})$-periodic of length 6 for every $\boldsymbol{u}$, where $h(\boldsymbol{w}) \neq \boldsymbol{0}$ by the bijectivity of h. Since h depends only on Λ, i.e. on A, but not on $\boldsymbol{u}$, we conclude that f is $h(\boldsymbol{w})$-periodic of length 6. $\square$

5. Configurations with four points

We define $\mathscr{P}_1$ to be the set containing the two elements

$$\{(1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 0), (0, 0, 0, 1)\}$$

and

$$\{(0, 1, 1, 1), (1, 0, 1, 1), (1, 1, 0, 1), (1, 1, 1, 0)\}.$$

We denote by $\mathscr{P}_2$ the set consisting of

$$\{(0, 0, 0, 0), (0, 0, 1, 1), (1, 0, 1, 0), (1, 1, 1, 1)\} \tag{16}$$

and the other 11 sets of four patterns which are generated from (16) by permutation of coordinates.

This section provides a proof for

Theorem 4. (PP) *is true for arbitrary* $f : \mathbb{Z}^k \to \{0, 1\}$ *and all configurations* A *with* $|A| \leqslant 4$ *except for the* A*'s with* $P_f(A) \in \mathscr{P}_1 \cup \mathscr{P}_2$.

Remark 8. It has been shown in Examples 1 and 2 of Section 3 that the exceptional sets of patterns in $\mathscr{P}_1 \cup \mathscr{P}_2$ of Theorem 4 are indeed exceptions to (PP).

The strategy in our proof of Theorem 4 is the following. By projection we define for each $\boldsymbol{u} \in \mathbb{Z}^k$ a function $f_{\boldsymbol{u}} : \mathbb{Z}^3 \to \{0,1\}$ which satisfies $P_{f_{\boldsymbol{u}}}(0,1,2,3) \subseteq P_f(A)$. If $|P_{f_{\boldsymbol{u}}}(0,1,2,3)| \leqslant 3$ for all $\boldsymbol{u}$, then Proposition 3 tells us exactly which patterns can be generated by $f_{\boldsymbol{u}}$'s that are not totally periodic, and this implies the desired result for f itself. If $|P_{f_{\boldsymbol{u}}}(0,1,2,3)| = 4$ for some $\boldsymbol{u}$, then Proposition 4 does the job. A considerable reduction in the proof of Proposition 4 is provided by Lemma 3, which helps to get rid of so-called blocking situations.

Let us call two vectors $(\varepsilon_1,\ldots,\varepsilon_n)$ and $(\delta_1,\ldots,\delta_n)$ in $\{0,1\}^n$ *complementary* if $\varepsilon_i \neq \delta_i$ for $1 \leqslant i \leqslant n$. We denote by $\mathscr{P}_3$ the set consisting of all sets with three vectors from $\{0,1\}^4$ containing either $(0,0,0,0)$ or $(1,1,1,1)$ and two other complementary vectors. We denote by $\mathscr{P}_4$ the set consisting of all sets with three vectors of $\{0,1\}^4$ containing $(0,0,0,0)$ and $(1,1,1,1)$ and one other vector. Clearly, $\mathscr{P}_3$ and $\mathscr{P}_4$ both have 14 elements.

Proposition 3. *Let $f : \mathbb{Z}^k \to \{0,1\}$ be a given function with $|P_f(0,n_1,n_2,n_3)| \leqslant 3$ for some positive integers $n_1 < n_2 < n_3 \leqslant k$. Then f is $A(\boldsymbol{e}_{n_1},\boldsymbol{e}_{n_2},\boldsymbol{e}_{n_3})$-periodic of length at most 6 except when $P_f(0,n_1,n_2,n_3) \in \mathscr{P}_3 \cup \mathscr{P}_4$, where the corresponding functions f have period dimension 2.*

Proof. Without loss of generality, let $k=3$, thus $n_i = i$ for $i=1,2,3$. We first assume that $|P_f(0,1,2,3)| \leqslant 2$. Then trivially $|P_f(0,n)| \leqslant 2$ for $n=1,2,3$. By Proposition 1, f is $\boldsymbol{e}_n$-periodic of length 2 for $n=1,2,3$. Lemma 1 implies total periodicity of length 2 which proves the assertion.

It remains to consider the case $|P_f(0,1,2,3)| = 3$. Let $A_i := E_3(0,1,2,3) \setminus E_3(i)$ for $1 \leqslant i \leqslant 3$. Applying Proposition 2 to a fixed A_i, it follows that either the period dimension of f is at least 2 with length of period at most 3, or we have one of the exceptional sets of patterns mentioned in Proposition 2. If we have the first situation for at least two configurations A_i and A_j, $1 \leqslant i < j \leqslant 3$, the two corresponding two-dimensional period lattices apparently do not lie in a common two-dimensional space. Therefore, Lemma 1 implies total periodicity of f of length at most $\mathrm{lcm}(1,2,3) = 6$. Hence we can assume that for at least two of the A_i's, we have an exceptional set of patterns as in Proposition 2. Without loss of generality let A_1 and A_2 have this property. Therefore

$$P_f(A_1) = P_f(0,2,3) = \{(\alpha_0,\alpha_2,\alpha_3),(\beta_0,\beta_2,\beta_3),(\gamma_0,\gamma_2,\gamma_3)\}, \tag{17}$$

say, for one of the exceptional sets of patterns of Proposition 2. This implies that

$$P_f(0,1,2,3) = \{(\alpha_0,\varepsilon_1,\alpha_2,\alpha_3),(\beta_0,\varepsilon_2,\beta_2,\beta_3),(\gamma_0,\varepsilon_3,\gamma_2,\gamma_3)\} \tag{18}$$

for some $\varepsilon_1, \varepsilon_2, \varepsilon_3 \in \{0,1\}$. It follows that

$$P_f(A_2) = P_f(0,1,3) = \{(\alpha_0, \varepsilon_1, \alpha_3), (\beta_0, \varepsilon_2, \beta_3), (\gamma_0, \varepsilon_3, \gamma_3)\}, \tag{19}$$

which, by our assumption, has to be an exceptional set of patterns as in Proposition 2. Checking each exceptional set of patterns (17) explicitly, we thus obtain specific values for $\varepsilon_1, \varepsilon_2, \varepsilon_3$ each time and thus finitely many sets (18). By doing this, we obtain up to exchange of 0 and 1 or permutation of coordinates the following sets of patterns:

$$\{(0,0,0,0),(0,0,1,1),(1,1,0,0)\}, \tag{20}$$

$$\{(0,0,0,0),(0,1,1,1),(1,0,0,0)\}, \tag{21}$$

$$\{(0,0,0,0),(0,0,1,1),(1,1,1,1)\}, \tag{22}$$

$$\{(0,0,0,0),(0,1,1,1),(1,1,1,1)\}, \tag{23}$$

$$\begin{aligned}&\{(0,0,0,0),(0,1,1,0),(1,1,0,1)\},\ \ \{(1,0,0,0),(0,1,0,0),(1,0,1,1)\},\\ &\quad\{(1,0,0,0),(0,1,0,0),(1,1,1,1)\},\end{aligned} \tag{24}$$

$$\{(0,1,0,0),(0,0,1,0),(1,0,0,1)\}. \tag{25}$$

The patterns in (20) and (21) belong to $\mathscr{P}_3$, (22) and (23) belong to $\mathscr{P}_4$. It is easy to show that none of the three sets of patterns in (24) equals $P_f(0,1,2,3)$ for any function f, and we exemplify this for $\{(0,0,0,0),(0,1,1,0),(1,1,0,1)\}$. Imbedding the first three coordinates in $\mathbb{Z}^2$, as we did in the proof of Proposition 2, we have

$$\begin{array}{ll}0 & \\ 0 & 0\end{array}\qquad\qquad \begin{array}{ll}1 & \\ 0 & 1\end{array}\qquad\qquad \begin{array}{ll}0 & \\ 1 & 1\end{array}$$

The second of these patterns can be extended to the right only by the last one, hence we obtain

$$\begin{array}{lll}1 & 0 & \\ 0 & 1 & 1\end{array}$$

but $(1,0)$ does not occur as a horizontal pattern. The final set of patterns we have to deal with is (25). The first three coordinates of these patterns are identical with the patterns of (14), and in the proof of Proposition 2 the corresponding functions were shown to be totally periodic of lenght 3 (in $\mathbb{Z}^2$). It is immediate that a function f with $P_f(0,1,2,3)$ as in (25) is $\boldsymbol{e}_3$-periodic of length 1. Altogether, Lemma 1 implies that such a function is totally periodic of length 3.

We have yet to show that the functions f with $P_f(0,n_1,n_2,n_3) \in \mathscr{P}_3 \cup \mathscr{P}_4$ have period dimension 2. In (20) and (22) both $\boldsymbol{e}_1$ and $\boldsymbol{e}_2 - \boldsymbol{e}_3$ are period vectors of f, since $P_f(E_3(0,1)) = P_f(E_3(2,3)) = 2$. Similarly, we find period vectors $\boldsymbol{e}_1 - \boldsymbol{e}_2$ and $\boldsymbol{e}_2 - \boldsymbol{e}_3$ in (21) and (23). In each case Lemma 1 can be used to verify the assertion of Proposition 3. This completes the proof. $\square$

We shall now present a method that will later on allow us to determine explicitly the coordinates of patterns under suitable conditions. For a set of patterns $(\varepsilon_{i,0}, \varepsilon_{i,1}, \dots, \varepsilon_{i,n-1}) \in \{0,1\}^n$ with $i = 1,2,3,\dots$, we speak of a *blocking situation with respect to the coordinate* ε_{i_0,j_0} for some $i_0 \geqslant 1$ and $0 \leqslant j_0 < n$, if $\varepsilon_{i,j_0} \neq \varepsilon_{i_0,j_0}$ for all $i \neq i_0$ and in the exceptional pattern $(\varepsilon_{i_0,0}, \varepsilon_{i_0,1}, \dots, \varepsilon_{i_0,n-1})$ the unique symbol ε_{i_0,j_0} occurs elsewhere, i.e. $\varepsilon_{i_0,j_1} = \varepsilon_{i_0,j_0}$ for some $j_1 \neq j_0$. If $A = \{\boldsymbol{a}_0, \boldsymbol{a}_1, \dots, \boldsymbol{a}_{n-1}\}$ is the corresponding configuration, then in a blocking situation $f(\boldsymbol{v}) = \varepsilon_{i_0,j_0}$ for some $\boldsymbol{v}$ implies $f(\boldsymbol{v} + \boldsymbol{a}_{j_1} - \boldsymbol{a}_{j_0}) = \varepsilon_{i_0,j_0}$. It follows that for every $\boldsymbol{v}$ there can be at most one value change in the sequence $(f(\boldsymbol{v} + m(\boldsymbol{a}_{j_1} - \boldsymbol{a}_{j_0}))_{m \in \mathbb{Z}}$.

Lemma 3. *Assume that we have a blocking situation in a set*

$$P_f(0,1,2,3) = \{(\varepsilon_{i,0}, \varepsilon_{i,1}, \varepsilon_{i,2}, \varepsilon_{i,3}) \colon i = 1,2,3,4\}$$

of four distinct patterns for some function $f\colon \mathbb{Z}^3 \to \{0,1\}$. *Then we have* $|P_f(E')| \leqslant |E'|$ *for a proper subset* $E' \subset E_3(0,1,2,3)$.

Proof. We make the assumption that

$$|P_f(E')| > |E'| \tag{26}$$

for all proper subsets $E' \subset E_3(0,1,2,3)$ and have to show that this is contradictory.

Since we have a blocking situation, we may assume without loss of generality (i.e. up to exchange of 0 and 1 or interchange of coordinates) that $\varepsilon_{1,0} = \varepsilon_{1,1} = 1$ and $\varepsilon_{2,0} = \varepsilon_{3,0} = \varepsilon_{4,0} = 0$. Looking at the first two coordinates of our four patterns in $P_f(0,1,2,3)$, we have by (26) without loss of generality

$$P_f(0,1,2,3) = \{(1,1,\varepsilon_{1,2},\varepsilon_{1,3}),(0,0,\varepsilon_{2,2},\varepsilon_{2,3}),(0,1,\varepsilon_{3,2},\varepsilon_{3,3}),(0,\varepsilon_{4,1},\varepsilon_{4,2},\varepsilon_{4,3})\}. \tag{27}$$

By (26) again, the pair $(\varepsilon_{1,2}, \varepsilon_{1,3})$ cannot match the pair $(\varepsilon_{3,2}, \varepsilon_{3,3})$, hence $\varepsilon_{1,2} \neq \varepsilon_{3,2}$ without loss of generality. If $\varepsilon_{1,2} = 0$ and $\varepsilon_{3,2} = 1$, then the two-dimensional notation for the first three coordinates (as used in the proof of Proposition 2) yields the corresponding patterns

$$\begin{matrix} 0 & \\ 1 & 1 \end{matrix} \qquad\qquad \begin{matrix} 1 & \\ 0 & 1 \end{matrix}$$

Continuing the second of these to the right, that is in the direction of $\boldsymbol{e}_1$, we have to use the first pattern in (27) since this is the only pattern with 1 in the first coordinate.

We obtain

$$\begin{array}{ccc} 1 & 0 & \\ 0 & 1 & 1 \end{array}$$

but the pattern $(1,0)$ does not occur in the first two coordinates of (27). Consequently, we must have $\varepsilon_{1,2}=1$ and $\varepsilon_{3,2}=0$. Then the first three coordinates of the third pattern of $P_f(0,1,2,3)$ look like

$$\begin{array}{cc} 0 & \\ 0 & 1 \end{array}$$

and can be extended to the left, i.e. in the direction of $-\boldsymbol{e}_1$, only by $(0,0,\varepsilon_{2,2})$, or by $(0,\varepsilon_{4,1},\varepsilon_{4,2})$ if $\varepsilon_{4,1}=0$. Since we do not have a $(1,0)$ in the first two coordinates, it follows that $\varepsilon_{2,2}=0$ or $\varepsilon_{4,2}=0$. We assume without loss of generality that $\varepsilon_{2,2}=0$. It is then an immediate consequence of (26) that $\varepsilon_{4,2}=1$ and $\varepsilon_{2,3}\neq\varepsilon_{3,3}$. Therefore, we either have

$$P_f(0,1,2,3)=\{(1,1,1,\varepsilon_{1,3}),(0,0,0,0),(0,1,0,1),(0,\varepsilon_{4,1},1,\varepsilon_{4,3})\} \tag{28}$$

or

$$P_f(0,1,2,3)=\{(1,1,1,\varepsilon_{1,3}),(0,0,0,1),(0,1,0,0),(0,\varepsilon_{4,1},1,\varepsilon_{4,3})\}. \tag{29}$$

Now we consider in both cases the triples generated by the first, second and fourth coordinates of the patterns, but visualize them by use of the earlier two-dimensional notation. In (28), the extension of the third pattern in the direction of $\boldsymbol{e}_1$ can only be done with the first pattern and yields

$$\begin{array}{ccc} 1 & \varepsilon_{1,3} & \\ 0 & 1 & 1 \end{array}$$

which, like before, implies $\varepsilon_{1,3}=1$. With (26), we conclude $\varepsilon_{4,1}\neq\varepsilon_{4,3}$. By symmetry, we may assume without loss of generality that $\varepsilon_{4,1}=0$ and $\varepsilon_{4,3}=1$, and we end up in case (28) with

$$P_f(0,1,2,3)=\{(1,1,1,1),(0,0,0,0),(0,1,0,1),(0,0,1,1)\}. \tag{30}$$

In (29), the third pattern cannot be extended in the direction of $-\boldsymbol{e}_1$ by use of the second pattern since we would obtain

$$\begin{array}{ccc} 1 & 0 & \\ 0 & 0 & 1 \end{array}$$

but $(1,0)$ is not among the first two coordinates of (27). Hence we have to use the last pattern where necessarily $\varepsilon_{4,1}=0$, and thus $\varepsilon_{4,3}=0$ by (26). Therefore, (29) finally leads to

$$P_f(0,1,2,3)=\{(1,1,1,\varepsilon_{1,3}),(0,0,0,1),(0,1,0,0),(0,0,1,0)\}. \tag{31}$$

All that is left to do is to show that the sets of patterns in (30) and (31) both are not realizable which means that none of them is the set of patterns for some function $f:\mathbb{Z}^3\to\{0,1\}$. Representing the first three coordinates of these two sets of patterns in the usual two-dimensional way, we have in both cases

$$\begin{array}{ll} 1 & \\ 1 & 1 \end{array} \qquad \begin{array}{ll} 0 & \\ 0 & 0 \end{array} \qquad \begin{array}{ll} 0 & \\ 0 & 1 \end{array} \qquad \begin{array}{ll} 1 & \\ 0 & 0 \end{array} \tag{32}$$

These patterns correspond to the planes parallel with $\Lambda(\boldsymbol{e}_1,\boldsymbol{e}_2)$. Apart from the trivial cases, by which we mean planes with constant values, the distribution of 0's and 1's on these planes generated by (32) may only look like

$$\begin{array}{cccccccc} & & & \vdots & \vdots & & & \\ & 0 & 0 & 0 & 1 & 1 & 1 & \\ \dots & 0 & 0 & 0 & 1 & 1 & 1 & \dots \\ & 0 & 0 & 0 & 1 & 1 & 1 & \\ & & & \vdots & \vdots & & & \end{array} \tag{33}$$

or

$$\begin{array}{ccccccc} & & & \vdots & & & \\ & 1 & 1 & 1 & 1 & 1 & \\ \dots & 1 & 1 & 1 & 1 & 1 & \dots \\ \dots & 0 & 0 & 0 & 0 & 0 & \dots \\ & 0 & 0 & 0 & 0 & 0 & \\ & & & \vdots & & & \end{array} \tag{34}$$

or

$$\begin{array}{cccccccc} & & & \vdots & \vdots & & & \\ & 0 & 0 & 0 & 1 & 1 & 1 & \\ & 0 & 0 & 0 & 1 & 1 & 1 & \\ \dots & 0 & 0 & 0 & 1 & 1 & 1 & \dots \\ & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ & 0 & 0 & 0 & 0 & 0 & 0 & \\ & & & \vdots & \vdots & & & \end{array} \tag{35}$$

Since $|P_f(0,1,2,3)|=4$, all patterns of (32) have to occur somewhere. Hence either some plane parallel with $\Lambda(\boldsymbol{e}_1,\boldsymbol{e}_2)$ is of type (35), or planes of type (33) and (34) occur simultaneously. The second situation is impossible: in case (30), we have always a 0 above

$$\begin{array}{ll} 0 & \\ 0 & 0 \end{array}$$

(in the direction of $\boldsymbol{e}_3$), and in case (31), there are only 1's above that pattern. Hence (33) can lie neither above nor below (34) somewhere.

Consequently, a plane of type (35) has to occur, and we have a subpattern

$$\begin{array}{lll} 0 & & \\ 0 & 1 & \\ 0 & 0 & 0 \end{array} \tag{36}$$

on some plane parallel with $\Lambda(\boldsymbol{e}_1, \boldsymbol{e}_2)$. Now look at the next plane in the direction of $\boldsymbol{e}_3$. In case (30), we obtain a pattern

$$\begin{array}{ll} 1 & \\ 0 & 1 \end{array}$$

In case (31), we have

$$\begin{array}{ll} 0 & \\ 1 & 0 \end{array}$$

However, both of these patterns cannot be found among the first three coordinates of (30) or (31). This concludes the proof of Lemma 3. □

Proposition 4. *Let $f : \mathbb{Z}^k \to \{0, 1\}$ be a given function with $|P_f(0, n_1, n_2, n_3)| = 4$ for some positive integers $n_1 < n_2 < n_3 \leqslant k$. Moreover, we assume that*

$$|P_f(E')| > |E'| \tag{37}$$

for all proper subsets $E' \subset E_3(0, n_1, n_2, n_3)$. Then

$$P_f(0, n_1, n_2, n_3) = \{(1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 0), (0, 0, 0, 1)\}$$

up to exchange of 0 *and* 1. *Moreover, f has period dimension* 2 *and is $\Lambda(\boldsymbol{e}_{n_1} - \boldsymbol{e}_{n_2}, \boldsymbol{e}_{n_3})$-periodic of length at most* 2 *up to interchange of coordinates.*

Proof. Without loss of generality, let $k = 3$, thus $n_i = i$ for $i = 1, 2, 3$. Let

$$P_f(0, 1, 2, 3) = \{(\varepsilon_{i,0}, \varepsilon_{i,1}, \varepsilon_{i,2}, \varepsilon_{i,3}) \colon \; i = 1, 2, 3, 4\},$$

say, with four distinct elements. First of all, we show that we cannot have all possible patterns $(0, 0)$, $(0, 1)$, $(1, 0)$, $(1, 1)$ in every subconfiguration $E' \subset E_3(0, 1, 2, 3)$ with $|E'| = 2$. Assuming the opposite, and using the notation $\bar{0} := 1$, $\bar{1} := 0$, we obtain for some $\varepsilon, \delta \in \{0, 1\}$

$$P_f(0,1,2,3) = \{(0,0,\varepsilon,\delta),(0,1,\bar{\varepsilon},\bar{\delta}),(1,0,\bar{\varepsilon},\bar{\delta}),(1,1,\varepsilon_{4,2},\varepsilon_{4,3})\}$$

by comparison of appropriate pairs. But already, we have a repetition in the patterns of the last two coordinates.

Therefore, we assume without loss of generality that on the subconfiguration $E_3(0,1)$, corresponding to the first two coordinates in our patterns of $P_f(0,1,2,3)$, we have at most three distinct patterns, hence by (37) exactly three patterns. This means that on setting $\varepsilon := \varepsilon_{1,0}$, $\delta := \varepsilon_{1,1}$, $\mu := \varepsilon_{1,2}$, $v := \varepsilon_{1,3}$, we may assume without loss of generality $\varepsilon_{2,0} = \varepsilon$, $\varepsilon_{2,1} = \delta$, $\varepsilon_{3,0} = \varepsilon$, $\varepsilon_{3,1} = \bar{\delta}$, and

$$\varepsilon_{4,0} = \bar{\varepsilon}, \varepsilon_{4,1} = \delta \tag{38}$$

or

$$\varepsilon_{4,0} = \bar{\varepsilon}, \varepsilon_{4,1} = \bar{\delta}. \tag{39}$$

By (37), we obtain immediately that $\varepsilon_{2,2} = \bar{\mu}, \varepsilon_{2,3} = \bar{v}$ in both cases. By symmetry, and by (37), the case (38) leads to

$$P_f(0,1,2,3) = \{(\varepsilon,\delta,\mu,v),(\varepsilon,\delta,\bar{\mu},\bar{v}),(\varepsilon,\bar{\delta},\mu,\bar{v}),(\bar{\varepsilon},\delta,\mu,\bar{v})\} \tag{40}$$

or

$$P_f(0,1,2,3) = \{(\varepsilon,\delta,\mu,v),(\varepsilon,\delta,\bar{\mu},\bar{v}),(\varepsilon,\bar{\delta},\mu,\bar{v}),(\bar{\varepsilon},\delta,\bar{\mu},v)\}. \tag{41}$$

From (39), we similarly obtain without loss of generality

$$P_f(0,1,2,3) = \{(\varepsilon,\delta,\mu,v),(\varepsilon,\delta,\bar{\mu},\bar{v}),(\varepsilon,\bar{\delta},\mu,\bar{v}),(\bar{\varepsilon},\bar{\delta},\bar{\mu},v)\} \tag{42}$$

or

$$P_f(0,1,2,3) = \{(\varepsilon,\delta,\mu,v),(\varepsilon,\delta,\bar{\mu},\bar{v}),(\varepsilon,\bar{\delta},\mu,\bar{v}),(\bar{\varepsilon},\bar{\delta},\mu,v)\}$$

or

$$P_f(0,1,2,3) = \{(\varepsilon,\delta,\mu,v),(\varepsilon,\delta,\bar{\mu},\bar{v}),(\varepsilon,\bar{\delta},\mu,\bar{v}),(\bar{\varepsilon},\bar{\delta},\bar{\mu},\bar{v})\}.$$

The second of the last three sets of patterns is easily seen to be equivalent with (41) by exchanging δ and μ. The last set of patterns is equivalent with (41): first exchange δ and $\bar{\delta}$ as well as v and $\bar{v}$, and then interchange δ and v. So we are left with the cases (40)–(42).

In (41), we have a blocking situation with respect to $\bar{\varepsilon}$, unless $\varepsilon = \delta = \bar{\mu} = v$. Since blocking contradicts (37) by Lemma 3, we conclude that indeed $\varepsilon = \delta = \bar{\mu} = v$. But there is also blocking with respect to $\bar{\bar{\delta}}$, unless $\varepsilon = \delta = \mu = \bar{v}$ which, once more by Lemma 3, must hold. This is impossible altogether.

In (42), we have a blocking situation only with respect to $\bar{\varepsilon}$, unless $\varepsilon = \bar{\bar{\delta}} = \bar{\mu} = v$. Again this has to hold, and on setting $\varepsilon = 0$ without loss of generality, we obtain

$$P_f(0,1,2,3) = \{(0,1,1,0),(0,1,0,1),(0,0,1,1),(1,0,0,0)\}. \tag{43}$$

We display part of two consecutive planes parallel with $\Lambda(\boldsymbol{e}_1, \boldsymbol{e}_2)$ containing the four entries of the third pattern in (43) in the following suggestive way, where each asterisk marks a yet unspecified position:

$$\begin{array}{ll} * & \\ 1 & * \\ \Uparrow & \\ 1 & \\ 0 & 0 \end{array}$$

Here the lower triple belongs to some plane parallel with $\Lambda(\boldsymbol{e}_1, \boldsymbol{e}_2)$, and the upper digit is an element of the plane above it in the direction of $\boldsymbol{e}_3$. In the upper plane, we have a unique continuation by the last pattern in (43), and this leads to a unique continuation to the right (i.e. in the direction of $\boldsymbol{e}_1$) by the first pattern of (43) in the lower plane, as shown in the left and right picture below:

$$\begin{array}{ll} 0 & \\ 1 & 0 \\ \Uparrow & \\ 1 & \\ 0 & 0 \end{array} \qquad\qquad \begin{array}{lll} 0 & * & \\ 1 & 0 & * \\ \Uparrow & & \\ 1 & 1 & \\ 0 & 0 & 1 \end{array}$$

The last pattern on the lower plane, however, is apparently not realizable because 11 cannot appear.

We are left with (40). We have blocking with respect to v, unless $\varepsilon = \delta = \mu = \bar{v}$. By Lemma 3, this must hold, and on setting $\varepsilon = 0$ without loss of generality, we obtain

$$P_f(0,1,2,3) = \{(1,0,0,0),(0,1,0,0),(0,0,1,0),(0,0,0,1)\}, \tag{44}$$

which proves the first part of Proposition 4.

We are now going to describe how the functions f satisfying (44) look like. Using three consecutive planes parallel with $\Lambda(\boldsymbol{e}_1, \boldsymbol{e}_2)$, the illustration introduced above shows that with the patterns of (44) each 1 is surrounded by 0's in the following way:

$$\begin{array}{ccc}
\ast & \ast & \\
0 & 0 & \ast \\
 & 0 & \ast \\
 & & \\
 & \Uparrow & \\
 & & \\
0 & 0 & \\
0 & 1 & 0 \\
 & 0 & 0 \\
 & & \\
 & \Uparrow & \\
 & & \\
\ast & 0 & \\
\ast & 0 & 0 \\
 & \ast & \ast
\end{array}$$

Let us first make the assumption that there is a pattern

$$\begin{array}{cc}
\ast & 1 \\
1 & \ast
\end{array} \tag{45}$$

somewhere, without loss of generality on some plane parallel with $\Lambda(\boldsymbol{e}_1, \boldsymbol{e}_2)$. Hence the surrounding 0's yield

$$\begin{array}{cccc}
 & \ast & \ast & \\
\ast & 0 & 0 & \ast \\
0 & 0 & 0 & \ast \\
 & 0 & \ast & \\
 & & & \\
 & \Uparrow & & \\
 & & & \\
 & 0 & 0 & \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
 & 0 & 0 & \\
 & & & \\
 & \Uparrow & & \\
 & & & \\
 & \ast & 0 & \\
\ast & 0 & 0 & 0 \\
\ast & 0 & 0 & \ast \\
 & \ast & \ast &
\end{array}$$

Starting with this pattern, we denote new 1's by $2, 3, 4, \ldots$ according to their order of construction and fill in corresponding 0's. By doing this, we obtain

$$
\begin{array}{cccccccc}
 & \ast & \ast & & & & & \\
\ast & \ast & 0 & 0 & & & & \\
0 & 0 & 0 & 10 & 0 & & & \\
 & 0 & 7 & \mathbf{0} & \mathbf{0} & 0 & & \\
 & & \mathbf{0} & \mathbf{0} & \mathbf{0} & 13 & 0 & \\
 & & & \mathbf{0} & 6 & 0 & 0 & \ast \\
 & & & & 0 & 0 & \ast & \ast \\
 & & & & & \ast & \ast &
\end{array}
$$

$$\Uparrow$$

$$
\begin{array}{cccccccc}
 & 0 & 0 & & & & & \\
0 & 0 & 9 & 0 & & & & \\
0 & 8 & 0 & \mathbf{0} & \mathbf{0} & & & \\
 & 0 & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} & & \\
 & & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} & 0 & \\
 & & & \mathbf{0} & \mathbf{0} & 0 & 12 & 0 \\
 & & & & 0 & 11 & 0 & 0 \\
 & & & & & 0 & 0 &
\end{array}
$$

$$\Uparrow$$

$$
\begin{array}{cccccccc}
 & \ast & \ast & & & & & \\
\ast & \ast & 0 & 0 & & & & \\
\ast & 0 & 0 & 4 & \mathbf{0} & & & \\
 & 0 & 5 & \mathbf{0} & \mathbf{0} & \mathbf{0} & & \\
 & & 0 & \mathbf{0} & \mathbf{0} & 2 & 0 & \\
 & & & 0 & 3 & 0 & 0 & \ast \\
 & & & & 0 & 0 & \ast & \ast \\
 & & & & & \ast & \ast &
\end{array}
$$

Since the pattern (45) is repeated along the diagonal in the direction of $\pm(\boldsymbol{e}_1 - \boldsymbol{e}_2)$, we have periodicity of length 2 along this strip of width 5. It is immediately seen that along the two boundaries of the strip, we must have sequences $\ldots, 0, 1, 0, 1, 0, 1, \ldots$ followed by diagonals of 0's. The position of the 1's on each of the new alternating diagonals may be chosen arbitrarily in one of the planes parallel with $\Lambda(\boldsymbol{e}_1, \boldsymbol{e}_2)$, and then the patterns in all other planes are uniquely determined. In this case, our assertion is proven.

Next we assume that a "horsejump" pattern

$$
\begin{array}{ccc}
\ast & \ast & 1 \\
1 & \ast & \ast
\end{array}
\tag{46}
$$

occurs somewhere, without loss of generality on some plane parallel with $\Lambda(\boldsymbol{e}_1, \boldsymbol{e}_2)$. Since we have dealt with the case (45) already, we may assume that such a pattern does not show up. Hence the surrounding 0's of (46) look like (we need only two planes)

$$
\begin{array}{ccccc}
 & & * & * & * \\
* & 0 & 0 & 0 & * \\
0 & 0 & \mathbf{1} & 0 & * \\
* & 0 & 0 & 0 & \\
\\
 & & \Uparrow & & \\
\\
 & & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & \mathbf{1} & 0 & 0 & 0 \\
0 & 0 & 0 & * &
\end{array}
$$

But here the two boldface 1's lie in diagonal position of type (45), and this case had been seen to satisfy our proposition.

By what we have shown so far, we may exclude any diagonal or horsejump position of 1's according to (45) or (46). This implies that any 1 is surrounded by 0's in the following way (again two planes are sufficient)

$$
\begin{array}{ccccc}
* & * & 0 & * & * \\
* & * & 0 & * & * \\
0 & 0 & 0 & 0 & 0 \\
* & * & 0 & * & * \\
* & * & 0 & * & * \\
\\
 & & \Uparrow & & \\
\\
\circ & 0 & \diamond & 0 & \circ \\
0 & 0 & 0 & 0 & 0 \\
\diamond & 0 & 1 & 0 & \diamond \\
0 & 0 & 0 & 0 & 0 \\
\circ & 0 & \diamond & 0 & \circ
\end{array}
$$

It is clear that we have to have 1's in the four ($\diamond$)-positions, and this in turn yields 1's in the ($\circ$)-positions. This pattern extends uniquely in the obvious way to a totally periodic function of length 2.

Proof of Theorem 4. Let $f:\mathbb{Z}^k \to \{0,1\}$ and $A = \{\mathbf{0}, \boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3\} \subseteq \mathbb{Z}^k$ be given. By Theorem 3 and Remark 5(ii) and (iii), we may assume that

$$|P_f(A)| = |A| = 4 \tag{47}$$

and

$$|P_f(A')| > |A'| \tag{48}$$

for all proper subsets $A' \subset A$. If $\dim_{\mathbb{R}}(\mathbb{R}\boldsymbol{a}_1 + \mathbb{R}\boldsymbol{a}_2 + \mathbb{R}\boldsymbol{a}_3) \leqslant 1$, the assertion follows from Theorem 2. Therefore, we make the further assumption that

$$\dim_{\mathbb{R}}(\mathbb{R}\boldsymbol{a}_1 + \mathbb{R}\boldsymbol{a}_2 + \mathbb{R}\boldsymbol{a}_3) \geqslant 2. \tag{49}$$

We define the two- or three-dimensional sublattice $\Lambda := \mathbb{Z}\boldsymbol{a}_1 + \mathbb{Z}\boldsymbol{a}_2 + \mathbb{Z}\boldsymbol{a}_3$ of $\mathbb{Z}^k$. Let $h : \mathbb{Z}^3 \to \Lambda$ be the linear map defined by $h(x_1, x_2, x_3) := x_1\boldsymbol{a}_1 + x_2\boldsymbol{a}_2 + x_3\boldsymbol{a}_3$ for $x_1, x_2, x_3 \in \mathbb{Z}$. By (49), the rank of h is at least 2. Given some $\boldsymbol{u} \in \mathbb{Z}^k$, we define the function $f_{\boldsymbol{u}} : \mathbb{Z}^3 \to \{0, 1\}$ by $f_{\boldsymbol{u}}(\boldsymbol{x}) := f(\boldsymbol{u} + h(\boldsymbol{x}))$ for $\boldsymbol{x} \in \mathbb{Z}^3$. By construction, we clearly have

$$P_{f_{\boldsymbol{u}}}(0,1,2,3) \subseteq P_f(A). \tag{50}$$

If $|P_{f_{\boldsymbol{u}}}(0,1,2,3)| = 4$ for some $\boldsymbol{u} \in \mathbb{Z}^k$, then we have by (50) and (47) that $P_{f_{\boldsymbol{u}}}(0,1,2,3) = P_f(A)$. Recall that we work up to exchange of 0 and 1. With (48), Proposition 4 yields $P_{f_{\boldsymbol{u}}}(0,1,2,3) = \{(1,0,0,0), (0,1,0,0), (0,0,1,0), (0,0,0,1)\}$, hence $P_f(A) = \{(1,0,0,0), (0,1,0,0), (0,0,1,0), (0,0,0,1)\}$, which means that $P_f(A) \in \mathscr{P}_1$, and we are done.

We may thus assume that

$$|P_{f_{\boldsymbol{u}}}(0,1,2,3)| \leqslant 3 \tag{51}$$

for all $\boldsymbol{u} \in \mathbb{Z}^k$. By Proposition 3, we know that either $f_{\boldsymbol{u}}$ is totally periodic of length at most 6, or $P_{f_{\boldsymbol{u}}}(0,1,2,3) \in \mathscr{P}_3 \cup \mathscr{P}_4$.

Now let us first assume that

$$P_{f_{\boldsymbol{u}_0}}(0,1,2,3) \in \mathscr{P}_3 \tag{52}$$

for some vector $\boldsymbol{u}_0 \in \mathbb{Z}^k$, without loss of generality

$$P_{f_{\boldsymbol{u}_0}}(0,1,2,3) = \{(0,0,0,0), (0,0,0,1), (1,1,1,0)\}$$

or

$$P_{f_{\boldsymbol{u}_0}}(0,1,2,3) = \{(0,0,0,0), (0,0,1,1), (1,1,0,0)\}.$$

For the first set of patterns, we have with (50) and (47) that $P_f(A') \leqslant 3$ for $A' := \{\boldsymbol{0}, \boldsymbol{a}_1, \boldsymbol{a}_2\}$, contradicting (48). Therefore only the second set of patterns has to be considered. Then $f_{\boldsymbol{u}_0}$ is $\Lambda(\boldsymbol{e}_1, \boldsymbol{e}_2 - \boldsymbol{e}_3)$-periodic, which we have seen already in the proof of Proposition 3 (cf. (20)). If there is another vector $\boldsymbol{u}_1 \in \mathbb{Z}^k$ such that $P_{f_{\boldsymbol{u}_1}}(0,1,2,3) \in \mathscr{P}_3 \cup \mathscr{P}_4$, but $P_{f_{\boldsymbol{u}_1}}(0,1,2,3) \neq P_{f_{\boldsymbol{u}_0}}(0,1,2,3)$, then it is immediately clear from (50) and (47) that

$$P_f(A) = \{(0,0,0,0), (0,0,1,1), (1,1,0,0), (1,1,1,1)\}. \tag{53}$$

Hence we have for any $\boldsymbol{u} \in \mathbb{Z}^k$ that $f_{\boldsymbol{u}}$ is either totally periodic of length at most 6, or it is one of the four subsets of (53) with three elements. But the four corresponding functions $f_{\boldsymbol{u}}$ are all $\Lambda(\boldsymbol{e}_1, \boldsymbol{e}_2 - \boldsymbol{e}_3)$-periodic of length 1 (cf. (20) and (22) in the proof of Proposition 3). Altogether we have shown that in case (52), there is a common two-dimensional lattice Γ, say, which is a sublattice of all period lattices $\Lambda_{f_{\boldsymbol{u}}}$ with $\boldsymbol{u} \in \mathbb{Z}^k$. A fortiori, this is true by Proposition 3, if $P_{f_{\boldsymbol{u}}}(0,1,2,3) \notin \mathscr{P}_3 \cup \mathscr{P}_4$ for all $\boldsymbol{u} \in \mathbb{Z}^k$. Hence in these situations $f|_{\boldsymbol{u}+\Lambda}$ is $h(\Gamma)$-periodic for every $\boldsymbol{u}$. Since the rank of h is at least 2, we know that $h(\Gamma) \neq \{\mathbf{0}\}$. Since h depends only on A and not on $\boldsymbol{u}$, we conclude that f is $h(\Gamma)$-periodic.

We are left with the case that

$$P_{f_{\boldsymbol{u}}}(0,1,2,3) \notin \mathscr{P}_3$$

for all $\boldsymbol{u} \in \mathbb{Z}^k$, but

$$P_{f_{\boldsymbol{u}_0}}(0,1,2,3) \in \mathscr{P}_4$$

for some $\boldsymbol{u}_0 \in \mathbb{Z}^k$, without loss of generality

$$P_{f_{\boldsymbol{u}_0}}(0,1,2,3) = \{(0,0,0,0),(0,0,0,1),(1,1,1,1)\}$$

or

$$P_{f_{\boldsymbol{u}_0}}(0,1,2,3) = \{(0,0,0,0),(0,0,1,1),(1,1,1,1)\}.$$

Like above, the first set of patterns would contradict (48). Hence we consider the second set of patterns, i.e. patterns with an even number of 1's. If this is the only element from $\mathscr{P}_4$ which is the set of patterns of $P_{f_{\boldsymbol{u}}}(0,1,2,3)$ for some $\boldsymbol{u} \in \mathbb{Z}^k$, then by the argument just used, we deduce that f is $h(\Gamma)$-periodic. If however, there is another vector $\boldsymbol{u}_1 \in \mathbb{Z}^k$ such that $P_{f_{\boldsymbol{u}_1}}(0,1,2,3) \in \mathscr{P}_4$, but $P_{f_{\boldsymbol{u}_1}}(0,1,2,3) \neq P_{f_{\boldsymbol{u}_0}}(0,1,2,3)$, then it is immediately clear from (50) and (47) that the 1's in the middle pattern are at the complementary positions so that

$$P_{f_{\boldsymbol{u}_1}}(0,1,2,3) = \{(0,0,0,0),(1,1,0,0),(1,1,1,1)\}$$

and we have again (53) by use of the corresponding argument above, or there is a 1 at the same position and a 1 in a different position, which corresponds to $P_f(A) \in \mathscr{P}_2$.

6. Limitations to periodicity principle (PP)

In Section 3, we constructed counterexamples to (PP) by way of slice functions. However, the functions were always periodic on all cosets of the lattice generated by the configuration A. In particular, $A \subset \mathbb{Z}^k$ did not generate the complete lattice $\mathbb{Z}^k$, i.e. $\mathbb{Z}\boldsymbol{a}_1 + \cdots + \mathbb{Z}\boldsymbol{a}_k \neq \mathbb{Z}^k$ for all $\{\boldsymbol{a}_1,\ldots,\boldsymbol{a}_k\} \subseteq A$. As a consequence of the results of Sections 4 and 5, we show

Theorem 5. *Let $f : \mathbb{Z}^k \to \{0,1\}$ be a given function for a positive integer k. Let $A = \{\mathbf{0},\boldsymbol{a}_1,\ldots,\boldsymbol{a}_n\} \subset \mathbb{Z}^k$ be a configuration for some positive integer $n \leqslant 3$. If $|P_f(A)| \leqslant |A|$, then f is periodic on every coset of the lattice $\Lambda(\boldsymbol{a}_1,\ldots,\boldsymbol{a}_n)$.*

Proof. By Theorems 3 and 4, the conditions of Theorem 5 imply periodicity of f except for the case where $P_f(A) \in \mathscr{P}_1 \cup \mathscr{P}_2$. Therefore, f is periodic on every coset of $\Lambda(\boldsymbol{a}_1, \dots, \boldsymbol{a}_n)$ unless

$$P_f(A) \in \mathscr{P}_1 \cup \mathscr{P}_2, \tag{54}$$

which we assume henceforth. Let $A = \{\mathbf{0}, \boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3\}$, say, and let Λ be any coset of the lattice $\Lambda(\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3)$, without loss of generality $\Lambda = \Lambda(\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3)$. We have to show that $f|_\Lambda$ is periodic.

If $\dim_{\mathbb{R}}(\mathbb{R}\boldsymbol{a}_1 + \mathbb{R}\boldsymbol{a}_2 + \mathbb{R}\boldsymbol{a}_3) \leqslant 1$, then Theorem 2 implies what is needed. We may thus assume that

$$\dim_{\mathbb{R}}(\mathbb{R}\boldsymbol{a}_1 + \mathbb{R}\boldsymbol{a}_2 + \mathbb{R}\boldsymbol{a}_3) \geqslant 2. \tag{55}$$

Let $h : \mathbb{Z}^3 \to \Lambda$ be the linear map with $h(x_1, x_2, x_3) := x_1\boldsymbol{a}_1 + x_2\boldsymbol{a}_2 + x_3\boldsymbol{a}_3$ for $x_1, x_2, x_3 \in \mathbb{Z}$. The function $f_0 : \mathbb{Z}^3 \to \{0, 1\}$, defined by $f_0(\boldsymbol{x}) := f(h(\boldsymbol{x}))$ for $\boldsymbol{x} \in \mathbb{Z}^3$, satisfies $P_{f_0}(0, 1, 2, 3) \subseteq P_f(A)$. We distinguish between $|P_{f_0}(0, 1, 2, 3)| = 4$ and $|P_{f_0}(0, 1, 2, 3)| \leqslant 3$. For $|P_{f_0}(0, 1, 2, 3)| = 4$, assumption (37) is satisfied by (54), and Proposition 4 implies that f_0 has period dimension 2. For $|P_{f_0}(0, 1, 2, 3)| \leqslant 3$, it follows from Proposition 3 that the period dimension of f_0 is at least 2. Since h is of rank greater or equal 2 by (55), we clearly have that $f|_\Lambda$ has period dimension at least 1 (cf. the proof of Theorem 4), i.e. $f|_\Lambda$ is periodic. This completes the proof of Theorem 5. $\square$

The following example due to Cassaigne [10] shows that Theorem 5 would be false for arbitrarily large configurations, even in case $k = 2$.

Example 3. For $k = 2$ we consider the functions f_1 and f_2 from Example 1 as well as their slice function f, say. Define

$$A := \{(x, y) \in \mathbb{Z}^2 \colon\ 0 \leqslant x \leqslant 6,\ 2|x,\ 0 \leqslant y \leqslant 3\},$$

hence $|A| = 16$. It is easy to see that $|P_f(A)| = 8$. Let $A' := A \cup \{(1, 0)\}$. Trivially $|P_f(A')| \leqslant 2|P_f(A)| = 16$ (and in fact $|P_f(A')| = 16$). Hence $|P_f(A')| < |A'|$. Clearly, A' generates $\mathbb{Z}^2$, but f is not periodic.

Cassaigne's example seems to indicate that convexity plays a role. We call a set $A \subseteq \mathbb{Z}^k$ *convex* if it contains all points $\boldsymbol{x} \in \mathbb{Z}^k$ which can be represented by $\boldsymbol{x} = \sum_{j=1}^{t} \lambda_j \boldsymbol{a}_j$ for some $\boldsymbol{a}_j \in A$ and positive real numbers λ_j satisfying $\sum_{j=1}^{t} \lambda_j = 1$. Obviously, a convex set $A \subseteq \mathbb{Z}^k$ is the intersection of $\mathbb{Z}^k$ with a convex set in $\mathbb{R}^k$. The next example demonstrates that convexity of the configuration A is not sufficient for periodicity if A does not have maximal dimension.

Example 4. For $k = 2$ we consider once more the functions f_1 and f_2 from Example 1, which are only vertically or horizontally periodic, respectively. It is easy to define a function $f : \mathbb{Z}^3 \to \{0, 1\}$ by setting $f(x, y, z) := f_i(x, y)$ with $i = i(z) \in \{1, 2\}$

such that f is not periodic. However, $|P_f(A)|=4$ for the convex set $A=\{(0,0),(1,0),(0,1),(1,1)\}$.

A convex set $A\subseteq\mathbb{Z}^k$ is called a *convex body* if A is not contained in a hyperplane of $\mathbb{R}^k$; this is in accordance with euclidean topology, where a convex body contains inner points. Rectangular blocks of suitable dimension are very special convex bodies. The subsequent example shows that even for configurations of this type the periodicity principle (PP) does not hold in dimension $k\geqslant 3$.

Example 5. Let $k\geqslant 3$ and let $m_i\geqslant 2$ $(1\leqslant i\leqslant k)$ be given integers. Let

$$A:=\{(a_1,\dots,a_k)\in\mathbb{Z}^k:\ 0\leqslant a_i<m_i\ (1\leqslant i\leqslant k)\}.$$

Hence A is a k-dimensional rectangular block with $|A|=M:=\prod_{i=1}^k m_i$. Define $f:\mathbb{Z}^k\to\{0,1\}$ by setting for $\boldsymbol{x}=(x_1,\dots,x_k)$

$$f(\boldsymbol{x}):=1\quad\Leftrightarrow\quad x_1=x_2=\cdots=x_{k-1}=0\ \text{or}\ x_2=m_2,\ x_3=\cdots=x_k=0.$$

It is easy to see that

$$|P_f(A)|=\frac{M}{m_k}+\frac{M}{m_1}+1.$$

Consequently, we have $|P_f(A)|\leqslant|A|$ for $m_k\geqslant 3$. But apparently f is not periodic.

Finally we provide an example of a convex body configuration A with $|A|=9$ for which periodicity does not follow.

Example 6. Let $f:\mathbb{Z}^3\to\{0,1\}$ be the function f of Example 5 with $k=3$ and $m_2=2$. Let

$$A:=\{(a_1,0,a_3)\in\mathbb{Z}^3:\ 0\leqslant a_1<2,\ 0\leqslant a_3<4\}\cup\{(0,1,0)\}.$$

Clearly, $|A|=|P_f(A)|=9$, but f is not periodic.

Two questions remain open in this section:

- Does (PP) hold in any dimension $k\geqslant 3$ for configurations A which are k-dimensional cubes of side length 2? These are the only blocks which are not covered by Example 5.
- Does (PP) hold for convex body configurations A with $5\leqslant|A|\leqslant 8$? Example 6 shows that this is not the case for $|A|\geqslant 9$. In the following section (Remark 9) we shall prove that it holds for $|A|\leqslant 4$.

7. The periodicity principle for convex bodies in $\mathbb{Z}^2$

We believe the following periodicity principle to be true.

Periodicity Principle (PPC). *Let $f:\mathbb{Z}^2\to\{0,1\}$ be a given function. Let $A\subset\mathbb{Z}^2$ be a non-empty finite convex body. If $|P_f(A)|\leqslant|A|$, then f is periodic.*

Theorem 2 implies that (PPC) can be formulated equivalently with "convex set" instead of "convex body". We have seen in Example 5 that (PPC) could not hold for dimensions $k \geqslant 3$. We recall that (PPC) covers the case of rectangular configurations in two dimensions.

Proposition 5. *If $A \subseteq \mathbb{Z}^2$ is a convex body, then A generates $\mathbb{Z}^2$.*

Proof. Without loss of generality $\mathbf{0} \in A$. Starting with $\mathbf{0}$ and then picking successively $\boldsymbol{a}_1, \boldsymbol{a}_2 \in A$ such that $\boldsymbol{a}_1, \boldsymbol{a}_2$ are linearly independent and $\{\mathbf{0}, \boldsymbol{a}_1\}$ as well as $\{\mathbf{0}, \boldsymbol{a}_1, \boldsymbol{a}_2\}$ are convex, we obtain a finite convex body

$$A' := \{\mathbf{0}, \boldsymbol{a}_1, \boldsymbol{a}_2\} \subseteq A.$$

Assume that $d := |\det(\boldsymbol{a}_1, \dots, \boldsymbol{a}_k)| > 1$. We define the parallelogram

$$P := \{\lambda_1 \boldsymbol{a}_1 + \lambda_2 \boldsymbol{a}_2\colon\ 0 < \lambda_j < 1\} \subset \mathbb{R}^2.$$

Clearly P has volume d. We define another parallelogram P' by setting

$$P' := \{\lambda_1 \boldsymbol{a}_1 + \lambda_2 \boldsymbol{a}_2\colon\ -1 < \lambda_j < 1\}.$$

Then P' is a $\mathbf{0}$-symmetric convex body in $\mathbb{R}^2$ with volume $4d$. Since $d > 1$ by assumption, Minkowski's theorem (cf. [19, Section 5, Theorem 1]) implies that P' contains a lattice point $\boldsymbol{x}_0 \in \mathbb{Z}^2 \setminus \{\mathbf{0}\}$. For a suitable subset $J \subseteq \{1, 2\}$, we consequently have

$$\boldsymbol{x}_1 := \boldsymbol{x}_0 + \sum_{j \in J} \boldsymbol{a}_j \in \bar{P},$$

where $\bar{P}$ denotes the closure of P. Since $\boldsymbol{x}_0 \neq \mathbf{0}$ and does not lie on the boundary of P', it follows that $\boldsymbol{x}_1 \notin A' \cup \{\boldsymbol{a}_1 + \boldsymbol{a}_2\}$. Let

$$P^* := \{\lambda_1 \boldsymbol{a}_1 + \lambda_2 \boldsymbol{a}_2\colon\ 0 \leqslant \lambda_j \leqslant 1,\ \lambda_1 + \lambda_2 \leqslant 1\}.$$

If $\boldsymbol{x}_1 \in P^*$, this contradicts the convexity of A'. If $\boldsymbol{x}_1 \notin P^*$, then $\boldsymbol{a}_1 + \boldsymbol{a}_2 - \boldsymbol{x}_1 \in P^*$, which again contradicts the convexity of A'. Our assumption must be wrong and therefore $d \leqslant 1$.

The linear independence of $\boldsymbol{a}_1, \boldsymbol{a}_2$ implies $d \neq 0$. Since d is an integer, we have $d = 1$. It follows by [19, Section 3, Theorem 7] that A' generates $\mathbb{Z}^k$, which proves Proposition 5.

For $k \geqslant 3$, Proposition 5 would be false: The set

$$S := \{(0,0,0), (1,0,0), (0,1,0), (1,1,2)\}$$

is a convex body in $\mathbb{Z}^3$, but apparently S does not generate $\mathbb{Z}^3$.

Theorem 6. (PPC) *holds for all configurations A with $|A| \leqslant 4$.*

Proof. This is an immediate consequence of Proposition 5 and Theorem 5.

Remark 9. For $|A| \leqslant 4$, (PPC) would even hold in any dimension: By Theorems 6, 3 and 4, we are left with the case $f : \mathbb{Z}^3 \to \{0,1\}$, where $A \subset \mathbb{Z}^3$ is a convex body with $|A| = 4$ and $P_f(A) \in \mathscr{P}_1 \cup \mathscr{P}_2$. Clearly A generates a three-dimensional sublattice Γ of $\mathbb{Z}^3$. By Propositions 3 and 4, f has one of at most two distinct two-dimensional period lattices on each coset of Γ. Hence there is a common period on all these cosets, which makes f periodic.

(PPC) implies that if $A \subset \mathbb{Z}^2$ is an $m \times n$ rectangle with positive integers m, n and $f : \mathbb{Z}^2 \to \{0,1\}$ satisfies $|P_f(A)| \leqslant mn$, then f is periodic (the cases $m = 1$ or $n = 1$ follow from Theorem 2). We shall prove this in case $m = 2$ in our paper [27]. The conjecture for rectangles has been made independently by Berthé and Vuillon [6].

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